MATH2050C Assignment 2

Deadline: Jan 21, 2025.

Hand in: Section 2.4 no. 17. Supp Problems no 1, 4, 5.

Section 2.4 no. 14, 15, 17.

Supplementary Problems

1. Prove the Nested Interval Property: Let $[a_n, b_n]$ satisfies $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \geq 1$. Show that there is $x \in \mathbb{R}$ such that $x \in [a_n, b_n]$ for all $n \geq 1$. Hint: Use the order completeness property.

2. Find the decimal representation of the numbers 0.502 and 1/7.

3. Show that there are infinitely many rational and irrational numbers lying between two distinct numbers.

4. Show that the cardinal number of any interval is equal to the cardinal number of $\mathbb{R}$.

5. Show that $|\mathbb{R}^2| = |\mathbb{R}|$. Recall that $\mathbb{R}^2 = \{(x, y) : \ x, y \in \mathbb{R}\}$.

6. A real number is called an algebraic number if it is a root of some equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with integral coefficients. Show that the set of all algebraic numbers is a countable set containing all rational numbers and numbers of the form $a^{1/k}, a > 0, k \geq 1$.

See next page for notes on real numbers.

The Real Number System II

Last time we ended by showing the existence of irrational numbers in $\mathbb{R}$. Now we study the density of rational and irrational numbers.

Proposition 2.1 Let $x > 0$, there is a unique $n \in \mathbb{N}$ such that $n - 1 \leq x < n$.

Proof. The set $E = \{k : \ x < k\}$ is nonempty according to the Archimedean Property. By the well-ordering principle of $\mathbb{N}$, E has a least element n in E. It means $n - 1$ does not belong to E, so $n - 1 \leq x$.

The well-ordering principle refers to the following self-evident fact: Every nonempty set E in $\mathbb{N}$ has a least element m in E, that is, $m \leq k, \ \forall k \in E$.

Applying the same argument to $F = \{k : x \leq k\}$, there is a unique $m \in \mathbb{N}$ such that $m - 1 < x \leq m$.

Proposition 2.2 For any real numbers $a, b, \ 0 < a < b$, there is a rational number and an irrational number strictly lying between a and b.

Immediately it implies there are infinitely many rational numbers and irrational numbers strictly lying between a and b. (Why?)

Proof. It suffices to show there is some $m/n \in (a, b)$. In fact, since $b-a > 0$, by the Archimedean Principle, there is some $n \in \mathbb{N}$ such that $b - a > 1/n$, that is, $n(b - a) > 1$, or $nb > na + 1$. Now, Proposition 2.1 ensures that there is some m such that $m - 1 \leq na < m$. In particular, $a < m/n$. As $nb > na + 1 \geq m - 1 + 1 = m$, we have $b > m/n$ too, done.

Applying what has been proved to the numbers $a/\sqrt{2}$ and $b/\sqrt{2}$, we get a rational number c between $a/\sqrt{2}$ and $b/\sqrt{2}$. Then $\sqrt{2}c$ is an irrational number lying between a and b.

Up to this point we have shown that an order-complete field $\mathbb{R}$ has elements called rational and irrational numbers both of which are "dense" inside $\mathbb{R}$. Every positive rational number is of the form p/q. By the usual division rule it could be represented as $a_0.a_1a_2\cdots$ where $a_0 \in \mathbb{N}\bigcup\{0\}$ and $a_k \in \{0, 1, 2, \cdots, 9\}$. It inspires to represent the irrational numbers in a similar manner. An algorithm enables us to do this is already hidden in Proposition 2.1.

Now we show that every real number has a decimal representation. The algorithm is: Let x be a positive number. First, find $a_0 \in \mathbb{N}\bigcup\{0\}$ such that $0 \leq x - a_0 < 1$. The existence of a_0 is guaranteed by Proposition 2.1. Then $0 \leq 10(x - a_0) < 10$. Next, we find $a_1 \in \{0, 1, 2, \cdots, 9\}$ to satisfy $a_1 \leq 10(x - a_0) < a_1 + 1$. Then $0 \leq 10(x - a_0) - a_1 < 1$ and $0 \leq 10[10(x - a_0) - a_1] < 10$. We find $a_2 \in \{0, 1, 2, \cdots, 9\}$ such that $a_2 \leq 10[10(x - a_0) - a_1] < a_2 + 1$, so $0 \leq 10[10(x - a_0) - a_1] - a_2 < 1$ and $0 \leq 10\{10[10(x - a_0) - a_1] - a_2\} < 10$. Repeating this process, we get $a_k, k \geq 1$, in $\{0, 1, 2, \cdots, 9\}$, such that

$$0 \leq a - a_0.a_1a_2\cdots a_k < \frac{1}{10^k}\ , \quad k \geq 1\ ,$$

where

$$a_0.a_1a_2\cdots a_k \equiv a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_k}{10^k}\ .$$

We conclude that every positive real number x can be associated to a set consisting of rational numbers $\{a_0, \ a_0.a_1a_2, \ a_0.a_1a_2a_3, \cdots\}$ such that x is the supremum of this set. We write x as

$a_0.a_1a_2a_3\cdots$ and call it the decimal representation of x. Similar representations can be achieved by replacing 10 by other natural numbers greater than 1 in the algorithm above. Binary ($n = 2$), ternary ($n = 3$), hexadecimal ($n = 16$) representations are in particular useful.

We make a degression to study infinite sets.

Each nonempty set A is assigned a symbol called its cardinal numbers denoted by $|A|$. It is equal to the number of elements when the set is a finite one. Recall some definitions.

First, two sets A and B have the same cardinal numbers iff there is a bijective map between them. Write $|A| = |B|$ when this holds. Next, $|A| \leq |B|$ if there is an injective map from A to B. It is clear that $|A| \leq |B|$ and $|B| \leq |C|$ implies $|A| \leq |C|$.

Proposition 2.3 Given two sets A and B. If there is a surjective map from B to A, then $|A| \leq |B|$.

Proof. Let this map be f. For each $a \in A$, the preimage $f^{-1}(a)$ is nonempty since f is surjective. Pick some $b \in f^{-1}(a)$ and denote it by $g(a)$. Then the map $a \mapsto g(a)$ sets up an injective map from A to B, done.

We state without proof the following fundamental result.

Schroder-Bernstein Theorem. For two sets A and B, $|A| \leq |B|$ and $|A| \leq |B|$ implies $|A| = |B|$.

A set is called a countable set if it is a finite set or its cardinal number is equal to $|\mathbb{N}|$. An infinite set is called uncountable if it is not countable.

Proposition 2.4. Every infinite set A satisfies $|A| \geq |\mathbb{N}|$.

Proof. It suffices to show an infinite set A must contain a countably infinite subset. The identity map from this subset to itself is an injective map from this set to A. Indeed, pick a_1 from A and then $A \setminus \{a_1\}$ is still an infinite set. So pick a_2 from it. The set $A \setminus \{a_1, a_2\}$ is still an infinite set. Keep doing this we can extract a countable subset $\{a_1, a_2, a_3 \cdots, \}$ from A.

Therefore, countable infinity is the "smallest infinity". Properties of countable sets are listed below.

Proposition 2.5

(a) Let $A_k, k \geq 1$, be countable sets. Then $\bigcup_k A_k$ is countable.

(b) For an infinite set A and a countable set C, $|A \bigcup C| = |A|$.

(c) For an infinite set A and a countable subset C, $|A \setminus C| = |A|$ provided $A \setminus C$ is an infinite set.

Proof. (a) Done in class using the "snake map".

(b) Pick $B = \{a_1, a_2, \cdots\}$ from A. This is possible from the proof of Proposition 2.4. Then we have the decomposition $A = A \setminus B \bigcup B$ and $A \bigcup C = A \setminus B \bigcup B \bigcup C$. Since B and $B \bigcup C$ are both countably infinite, there is a bijective map between them. On the other hand, the identity map is a bijective map from $A \setminus B$ to itself. Putting these two maps together, we get a bijective map between A and $A \bigcup C$.

(c) Use (b) by observing $A = A \setminus C \bigcup C$.

Proposition 2.6 $|(0,1)| > |\mathbb{N}|$. In other words, there is an injective map from $\mathbb{N}$ to $(0,1)$ but there is no bijective map between $(0,1)$ and $\mathbb{N}$.

Proof. Every $x \in (0,1)$ has decimal representation $0.a_1a_2a_3\cdots$ where $a_k \in \{0,1,\cdots,9\}$. Let the set of decimal representations be D. It can be shown that $\varphi : x \mapsto 0.a_1a_2a_3\cdots$ is injective from $(0,1)$ to D. However, it is not surjective. Let $M = \{0.a_1a_2a_3\cdots : \ a_k = 9, \ \forall k \geq k_0 \text{ for some } k_0\}$. Then φ maps $(0,1)$ bijectively to $D \setminus M$. It is not hard to show that M is countable. By Proposition 2.5(c), $|D| = |D \setminus M|$. Next, we claim $|D| > |\mathbb{N}|$. This is done by the famous argument of Cantor, see Theorem 2.5.5 in our text book. It follows that $|(0,1)| = |D \setminus M| = |D| > |\mathbb{N}|$.

Proposition 2.7 $|\mathbb{R}| = |(0,1)|$.

Proof. The map $\tan \pi(x - 1/2)$ is bijective from $(0,1)$ to $\mathbb{R}$, done.

As an exercise, you are asked to show the cardinal number of any interval is equal to that of $\mathbb{R}$. It is natural to ask, is there a set A whose cardinal number lying strictly between the set of natural numbers and the set of real numbers, that is, $|\mathbb{N}| < |A| < |\mathbb{R}|$? This is called the continuum hypothesis. It is known that it is independent of axiomatic set theory, the foundation of mathematics used today. It means that the continuum hypotheses can never be proved or disproved!